Symmetry properties of the large-scale solar magnetic field

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We investigate symmetry properties of the solar magnetic field in the framework of the linear Parker's migratory dynamo model. We show that the problem can be mapped onto the well-known quantum-mechanical double-well problem. Using the WKB approximation, we construct analytical solutions for the dipole and quadrupole configurations of the generated magnetic field. Our asymptotic analysis within the equatorial region indicates the existence of an additional weak dynamo wave which violates the Hale's polarity law. We estimate the spatial decay decrement of this wave. We also calculate explicitly the splitting of the eigenvalues corresponding to the dipole and quadrupole modes. The real part of the dipole eigenvalue is shown to exceed the quadrupole one. A sufficiently long time after generation the dipole mode should dominate the quadrupole one. The relevant observational evidences of the properties obtained are discussed.

1 Introduction

The problem of solar magnetic activity is an old and intriguing subject. A great amount of observational data have been accumulated since the first half of 17th century. A lot of efforts have been made to explain theoretically these remarkable data. However, the problem still bears many unanswered questions. Among them, the problem of symmetry of the solar magnetic field exists. This problem is visualized as a symmetry violation in some tracers of the solar activity at low latitudes in certain periods of times such as the Maunder Minumum (Harvey, 1992; Ribes and Nesme-Ribes, 1993).

It is clear now that solar magnetic field generation is connected with a turbulent motion of the differentially rotating electrically conducting solar plasma. The full set of the corresponding magnetohydrodynamic equations is too complicated to handle analytically. However, a significant simplification arises if we limit ourselves to considering the large-scale magnetic structure only. The equations describing the large-scale magnetic field in a thin convective shell were obtained by Parker on phenomenological grounds back in 1955. These equations have been intensively investigated for almost

fifty years. A more solid basis of describing mean magnetic field was provided by Steenbeck, Krause and Rädler (1966) who formally derived the equations of mean-field magnetohydrodynamics. The Parker equations follow from the general equations of mean-field magnetohydrodynamics being, however, only the leading approximation with respect to the short length of a dynamo wave. For a more consistent treatment of the dynamo problem, the next-to-leading approximation is required and the Parker equations must be slightly modified (see *e.g.* the recent paper of Galitski and Sokoloff, 1999).

There is no doubt that the large-scale magnetic field observed on the today's Sun corresponds to the steady regime. Recent investigations of the Parker equations in the non-linear case have revealed the following three features of the steadily oscillatory solutions (see, *e.g.*, Meunier *et al.*, 1997; Bassom *et al.*, 1999; Kuzanyan and Sokoloff, 1996). Firstly, the structure of the steady state solution is not very sensitive to the explicit form of the nonlinearity introduced into the mean-field equations. Secondly, the spatial profile of the dynamo wave in the steady regime may retain some qualitative features of the one for the kinematic problem. Thirdly, the frequency of the magnetic field oscillations coincides in the main approximation with the imaginary part of the eigenvalue of the linear problem. Thus, it is quite reasonable to start up with a relatively simple linear case in which the equations may allow for an analytical solution. Moreover, the kinematic problem is a first step to approach the nonlinear case.

In the kinematic approximation, the mean-field equations reduce to an eigenvalue problem for a linear differential operator. Formally, this operator is quite similar to the standard Hamiltonian in non-relativistic quantum mechanics. The diffusion terms play the role of kinetic energy, while the alpha-effect (helicity coefficient) and the differential rotation play the role of a two-component external potential. It has been found fruitful to apply some well-established quantum-mechanical approaches to the problem at hand (e.g., Sokoloff *et al.*, 1983, Ruzmaikin *et al.*, 1990). Using the WKB approximation, Kuzanyan and Sokoloff (1995) and later Galitski and Sokoloff (1998, 1999) have obtained asymptotic analytical solutions of the dynamo problem in the framework of the Parker model. The solution describing the wave of magnetic activity, the so-called dynamo wave, was built up of the three parts (dynamo waves): a strong dynamo wave propagating towards the equator in the main spatial region, a relatively weak dynamo wave propagating in the subpolar region polewards and an extremely weak dynamo wave reflected from the pole and decaying exponentially propagat-

ing equatorwards. The former two waves are known from the observations of the Sun (Makarov and Sivaraman, 1983), while the latter wave predicted theoretically has not yet been observed probably due to its weakness.

A major assumption adopted in the works cited above was that the sources of magnetic field generation in the southern and northern hemispheres were well separated and the generation took place in different hemispheres absolutely independently. Certainly, this assumption is not appropriate when considering the symmetry problem in which the interplay between the dynamo waves in different hemispheres is the key factor. Moreover, the WKB approximation breaks down in the very vicinity of the equator as shown below.

In the present paper, we investigate the symmetry problem by solving the dynamo equations in the Parker's model. We investigate the equatorial region and show that the wave of magnetic activity generated in a given hemisphere does not vanish at the equator but rolls over it propagating ahead in the other hemisphere and decaying with propagation. Due to this fact, there is a weak interaction between the dynamo waves in different hemispheres which results in a small splitting of the eigenvalues corresponding to different symmetry configurations of the global magnetic field. We estimate the spatial decay rate of this straying dynamo wave. We also calculate the splitting explicitly and find that the growth rate for the dipole configuration exceeds the one for the quadrupole configuration in the framework of the Parker's model. Notice, that the asymptotic method used enables us to estimate just the asymptotic order of magnitude of this eigenvalue splitting and we can hardly calculate the exact numerical coefficient by this method.

2 Basic Equations

In the present section we describe the model and establish the necessary notations. Since, the current paper is based heavily on our previous work, we highlight the main result only briefly omitting all intermediate calculations. We refer an interested reader to the original paper of Kuzanyan and Sokoloff (1995) [see also Galitski and Sokoloff (1999)] for a more exhaustive presentation.

The starting point is the mean-field equations derived by Krause and Rädler (1980):

$$\frac{\partial \mathbf{B}}{\partial \mathbf{t}} = \nabla\times(\alpha\mathbf{B}) + \nabla\times(\mathbf{v}\times\mathbf{B}) + \beta\Delta\mathbf{B}, \tag{1}$$

where $\mathbf{B}$ and $\mathbf{v}$ are the large-scale (mean) magnetic and velocity fields correspondingly, α is the helicity coefficient and β is the turbulent diffusivity.

In the linear approximation the time dependent part can be factored out trivially:

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r})\mathrm{e}^{\gamma t}, \tag{2}$$

where γ is an imaginary parameter to be found.

We consider the $\alpha\Omega$-dynamo model and suppose that magnetic field generation occurs in a thin spherical shell corresponding to the convective shell of the Sun. We also suppose that the differential rotation is more intensive than the mean helicity and that the radial gradient of the mean angular velocity $G = \frac{1}{r}\frac{\partial \Omega}{\partial r}$ weakly depends on latitude θ.

To treat the problem, it is convenient to present the axisymmetric magnetic field as a superposition of the toroidal and poloidal components and express the latter component as follows: $\mathbf{B}_p = \nabla\times\,(0,\,0,\,A)$ (we use the spherical system of coordinates and measure off the latitude from the equator), where $\mathbf{A}$ a the vector potential.

After averaging Eq.(1) over the shell, one obtains

$$\gamma A = \alpha(\theta) B + \frac{d}{d\theta}\left[\frac{1}{\cos\theta}\frac{d}{d\theta}(A\cos\theta)\right], \tag{3}$$

$$\gamma B = -DG(\theta)\frac{d}{d\theta}(A\cos\theta) + \frac{d}{d\theta}\left[\frac{1}{\cos\theta}\frac{d}{d\theta}(B\cos\theta)\right]. \tag{4}$$

Note, that all quantities have been rescaled which leaded to the simple dimensionless form of the equations. Parameter D is the dimensionless dynamo number which is supposed to be negative and numerically large. Functions $\alpha(\theta)$ and $G(\theta)$ are measured in units of their maximum values and they are certainly unknown explicitly. Observations of the solar magnetic activity and helioseismological data (see *e.g.* Schou et al., 1998) enable us to estimate them with a good accuracy (*e.g.*, Belvedere et al., 2000). In what follows, we will suppose $\alpha(\theta)$ and $G(\theta)$ to be reasonable smooth functions subject to the following symmetry constraints

$$\alpha(\theta) = -\alpha(-\theta), \quad \text{and} \quad G(\theta) = G(-\theta).$$

One of the advantages of the asymptotic method we apply is that the explicit knowledge of the functions is not required and the final results can be expressed in a quite general form. To make explicit estimates, we use the following simple form of the functions: $\alpha(\theta) = \sin\theta$ and $G = 1$.

It is convenient to rewrite Eqs.(3, 4) in the following symbolic form:

$$\hat{\mathcal{H}}\vec{h} = \gamma\vec{h}, \tag{5}$$

where we introduced a two-component complex function

$$\vec{h}(\theta) = \begin{pmatrix} A(\theta) \\ B(\theta) \end{pmatrix} \tag{6}$$

and a matrix differential operator $\hat{\mathcal{H}}$ which is well-defined by Eqs.(3, 4):

$$\hat{\mathcal{H}} = \begin{pmatrix} \Delta & \alpha(\theta) \\ & \\ -DG(\theta)\frac{d}{d\theta}\cos\theta & \Delta \end{pmatrix}, \tag{7}$$

where a notation for the azimuthal part of the Laplacian is introduced for brevity $\Delta = \frac{d}{d\theta}\frac{1}{\cos\theta}\frac{d}{d\theta}\cos\theta$. Note that operator (7) is nonhermitian and, thus, its eigenvalues are complex in general. The adjoint operator has the following form

$$\hat{\mathcal{H}}^\dagger = \begin{pmatrix} \Delta & DG(\theta)\frac{d}{d\theta}\cos\theta \\ & \\ \alpha(\theta) & \Delta \end{pmatrix}. \tag{8}$$

Note that operators $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^\dagger$ are defined on the segment $\theta \in [-\pi/2, \pi/2]$.

The main idea of the WKB solution obtained in the above cited papers was to expand both the eigenvectors and eigenvalues into an asymptotic series in terms of parameter $\varepsilon = |D|^{-1/3}$, which was supposed to be small. Note, that magnetic field generation occurs only due to the helicity and differential rotation mechanisms [functions $\alpha(\theta)$ and $G(\theta)\cos(\theta)$]. Remarkably, the two physically different mechanisms collapse into one universal function $\tilde{\alpha}(\theta) = \alpha(\theta)G(\theta)\cos(\theta)$ in the framework of the asymptotic solution. This function is relatively small near the equator and the poles. The magnetic field appears in these regions mainly due to the diffusion from middle latitudes where the generation is efficient. The solution was obtained under the assumption that the magnetic field generated in different hemispheres independently. This is true if the interplay between the southern and northern dynamo waves is negligible.

In the framework of the WKB approach, we present the solution $\vec{n}$ in the northern hemisphere as a product of a fast oscillating exponent and a

slowly varying amplitude $\vec{n}_0$:

$$\vec{n}(\theta) = \vec{n}_0(\theta)\exp\left[\frac{iS(\theta)}{\varepsilon}\right]. \tag{9}$$

The eigenvalues are seeking in the form of the asymptotic series

$$\gamma = \varepsilon^{-2}\Gamma_0 + \varepsilon^{-1}\Gamma_{1,n}, \tag{10}$$

where integer index n parameterizes the discrete eigenvalues.

Using the WKB approximation technique, one can obtain the following explicit expression for the amplitude

$$\vec{n}_0(\theta) = \frac{1}{\cos\theta}\begin{pmatrix} \Gamma_0 + k(\theta)^2 \\ i\varepsilon^{-2}k(\theta)\cos\theta \end{pmatrix}\sigma(\theta), \tag{11}$$

where $k(\theta) = dS(\theta)/d\theta$ which satisfies the following Hamilton-Jacobi equation:

$$\left[\Gamma_0 + k^2(\theta)\right]^2 - i\tilde{\alpha}(\theta)k(\theta) = 0, \tag{12}$$

where $\tilde{\alpha} = \alpha(\theta)G(\theta)\cos\theta$. This algebraic equation possesses four branches of roots. It is impossible to construct a smooth solution using any individual branch. Such a solution appears only by matching two different branches. A continuous crossover from one branch to the other is possible only for some unique values of the spectral parameter. This condition pins the value of γ and determines the desired spectrum:

$$\Gamma_0 = \frac{3}{2^{8/3}}\,\tilde{\alpha}_{\max}^{2/3}\,\mathrm{e}^{i\frac{\pi}{3}} \tag{13}$$

and

$$\Gamma_{1,n} = 3ik'(\theta')\left[n + 1/2\right], \quad n = 0, 1, 2, ..., \tag{14}$$

where θ' is the matching point which is the point of maximum of function $\tilde{\alpha}(\theta)$. The leading mode corresponds to the value $n = 0$. Function $\sigma(\theta)$ in Eq.(11) has the following explicit form:

$$\sigma_n(\theta) = \exp\left\{\int \frac{\Gamma_{1,n} - ik'\left(1 + \dfrac{2k^2}{\Gamma_0 + k^2}\right)}{2ik + \dfrac{\hat{\alpha}}{2(\Gamma_0 + k^2)}}d\theta\right\}. \tag{15}$$

Equations (9—15) determine the asymptotic solution completely. We use these explicit formulae throughout the paper.

3 Equatorial region

The crucial assumption of the WKB approximation is that the amplitude of the eigenvector varies in space much slower than the exponential factor [see Eq.(9)]. This implies:

$$\frac{d\vec{n}_0}{d\theta} \ll \frac{1}{\varepsilon}\vec{n}_0.$$

One can easily check that in the vicinity of the equator $\theta \ll 1$, the following estimate holds:

$$\frac{d\vec{n}_0}{d\theta} \sim \frac{1}{\theta}\vec{n}_0.$$

Thus, in domain $\theta < \varepsilon$, the applicability of the WKB approximation becomes questionable.

Note, that at $\theta = 0$, two different branches $k(\theta)$ merge. In the WKB theory, such a point is called "turning point". It is more rule than exception that a WKB solution diverges at a turning point. Using explicit expressions (9—15) we, indeed, observe that out solution diverges at $\theta = 0$. It is possible to show that

$$\sigma(\theta) \approx \theta^{-1/4}, \quad \text{for } \theta \to 0.$$

Moreover, one can see that $k'(\theta) \sim \theta^{-1/2}$ diverges as well.

Note, that these infinities have no physical meaning and appear due to unjustified approximations. The true solution is indeed a smooth function everywhere including the equator. This can be easily seen by expanding equations (3, 4) in the vicinity of $\theta = 0$:

$$\gamma A(\theta) = \alpha'(0)\theta B(\theta) + A''(\theta), \tag{16}$$

$$\gamma B(\theta) = -DA'(\theta) + B''(\theta). \tag{17}$$

In these equations γ plays the role of an independent external parameter. In the region $\theta < \varepsilon$, the solution can be written as a superposition of two waves

$$\begin{pmatrix} A(\theta) \\ B(\theta) \end{pmatrix} = \begin{pmatrix} A_1 + \delta A_1(\theta) \\ B_1 + \delta B_1(\theta) \end{pmatrix} \mathrm{e}^{-\sqrt{\gamma}\theta} + \begin{pmatrix} A_2 + \delta A_2(\theta) \\ B_2 + \delta B_2(\theta) \end{pmatrix} \mathrm{e}^{\sqrt{\gamma}\theta}, \tag{18}$$

where $A_{1,2}$ and $B_{1,2}$ are some constants to be determined by matching of (18) with the solution in the main domain. The θ-dependent corrections to the amplitudes in Eq.(18) can be easily found explicitly. It is possible to show, that for $\theta \lesssim \varepsilon$ these corrections are small and can be safely neglected in the leading approximation. Solution (18) is a perfectly smooth function.

We denote the solution in the northern hemisphere as $\vec{n}(\theta)$ and in the southern hemisphere as $\vec{s}(\theta)$. Let us note, that the first term in Eq.(18) describes a plane wave propagating towards the equator, while the second one describes the wave propagating polewards and decaying exponentially. The first term can be matched with the WKB solution in the northern hemisphere $\vec{n}(\theta)$. The only way to match the second one is to suppose that the wave generated in the southern hemisphere $\vec{s}(\theta)$ penetrates to the northern hemisphere. In the vicinity of the equator the exponential factors in Eq.(18) and in the WKB solution coincide. Equating the WKB eigenvector $\vec{n}_0$ with the first exponential term coefficient in (18) and $\vec{s}_0$ with the second one, we can express the coefficients $A_{1,2}$ and $B_{1,2}$ through the WKB parameters.

Note that there are four different branches of roots $k(\theta)$ of Eq.(12). Two of them describe the dynamo wave in the main domain. As it was shown by Galitski and Sokoloff (1999), the third branch describes the wave reflected from the pole. One can check that the fourth branch left is the only possible candidate to describe the wave propagating over the equator. Only this branch decays all the way down to $\theta = -\pi/2$.

Using the asymptotic expansions in the WKB solution for the case $\theta \ll 1$, we obtain:

$$\vec{n} = \begin{pmatrix} a\,\mathrm{e}^{(i\pi/12)}\,\theta_1^{(1/4)} \\ \varepsilon^{-2} b\,\mathrm{e}^{(i\pi/6)}\,\theta_1^{(-1/4)} \end{pmatrix} \exp\left(-\sqrt{\gamma}\theta\right) \tag{19}$$

and the adjoint solution in the southern hemisphere:

$$\vec{s}^a(\theta) = \begin{pmatrix} \varepsilon^{-2} b\,\mathrm{e}^{(i\pi/12)}\,\theta_1^{(-1/4)} \\ a\,\mathrm{e}^{(-i\pi/3)}\,\theta_1^{(1/4)} \end{pmatrix} \exp\left(-\sqrt{\gamma^*}\theta\right), \tag{20}$$

where $\theta_1 \sim \varepsilon$ is the matching point and the following real constants have been introduced for brevity:

$$a = 2^{4/3} 3^{1/4} \sqrt{\tilde{\alpha}'(0)}\tilde{\alpha}_{\max}^{2/3} \tag{21}$$

and

$$b = 3^{1/2} 2^{(-4/3)}\,\tilde{\alpha}_{\max}^{1/3} \tag{22}$$

Now, we can estimate the decrement of spatial decay of the dynamo wave straying into the other hemisphere. Indeed, the characteristic latitude of its propagation ahead is $1/\sqrt{\mathrm{Re}\gamma}$. For the case $D = -10^3$, or $\varepsilon = 0.1$, we can estimate its leading order approximation using formulae (10) and (13) as 0.28, or 16°. Notice that for $D = -10^4$ this value is of order 7°. These

numbers look quite reasonable in view of the observational data obtained by Harvey (1992). She found the reversed polarity active regions that violate Hale's polarity law. Such regions appear in the final phase of the solar cycle mainly at low latitudes, near the solar equator. These results give signatures of the straying dynamo wave propagating from the counterpart hemisphere.

4 Symmetry properties

To formalize the subsequent calculations, let us introduce the following symmetry operator $\hat{P}$ which we define as follows:

$$\hat{P}f(\theta) = f(-\theta).$$

This operator has the following obvious eigenvalues (parity):

$$p = \pm 1.$$

Acting by this operator on the both sides of Eq.(1) and taking into account that $\hat{P}\alpha(\mathbf{r}) = -\alpha(\mathbf{r})$ and $\hat{P}\mathbf{v}(\mathbf{r}) = \mathbf{v}(\mathbf{r})$, we see that the operator on the right-hand side commutes with $\hat{P}$. Thus, its eigenfunctions, *i.e.* magnetic field $\mathbf{B}$, can be classified by the parity index p. Value $p = +1$ corresponds to the quadrupole solution, while value $p = -1$ corresponds to the dipole one. Eigenvalues γ corresponding to different parities do not coincide.

Since, Eqs.(3.4) [or equivalently Eq.(5)] follow from Eq.(1), they inherent the symmetry properties of the initial equations. Let us note that these reduced equations involve a toroidal component of the physical magnetic field and a component of the gauge field. These two fields have opposite parities. Thus, when dealing with composite objects like (6) which contain the both fields, it is necessary to take into account this difference. Let us introduce the following unitary matrix:

$$\hat{U} = \hat{U}^{\dagger} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{23}$$

The dipole solution satisfies the following condition

$$\hat{P}\hat{U}\vec{d} = \vec{d} \tag{24}$$

and consequently the quadrupole one is defined by

$$\hat{P}\hat{U}\vec{q} = -\vec{q}. \tag{25}$$

The corresponding equations are written as

$$\hat{\mathcal{H}}\vec{d} = \gamma_d\,\vec{d} \quad \text{and} \quad \hat{H}\vec{q} = \gamma_q\,\vec{q}. \tag{26}$$

We also define the eigenvectors for the adjoint operator (8) as

$$\hat{\mathcal{H}}^{\dagger}\vec{d}^{a} = \gamma_d^{*}\,\vec{d}^{a} \quad \text{and} \quad \hat{\mathcal{H}}^{\dagger}\vec{q}^{a} = \gamma_q^{*}\vec{q}^{a}. \tag{27}$$

Since the operator is nonhermitian, the eigenvectors for the mutually adjoint operators do not coincide. Let us note here that the eigenvalues may exist only in the form of complex conjugated pairs. Formally, this follows from the fact that operator $\hat{\mathcal{H}}$ as well as the magnetic field are real. Below we are mainly interested in the structure of eigenfunctions. However, the splitting of the eigenvalues is of some interest as well.

To find the splitting of the eigenvalues, we make use of a method used to solve a well-known quantum-mechanical problem. Namely, we observe that the problem at hand is very similar to the double-well problem in quantum mechanics. In the latter, a particle in a symmetric one-dimensional potential is studied. The potential consists of two quantum wells separated by a high barrier. If one neglects the possibility of the penetration through the barrier the eigenvalues are degenerated and they can be calculated in a well (say in the right one) with the help of the WKB approximation. The eigenfunctions (wave functions) decay exponentially far from the well. If one takes into account a finite probability of the barrier penetration, the degeneracy is lifted and the energy levels split into pairs corresponding to the symmetric and antisymmetric solutions. The quantity of interest is the energy difference between the two lowest eigenstates, which corresponds to the tunneling rate through the double well barrier. The quantity is often small and difficult to calculate numerically, especially when the potential barrier between the two wells is large. However, using the WKB eigenfunctions obtained for each quantum wells, it is easy to construct approximate symmetric and antisymmetric solutions explicitly. The subsequent calculation of the tunneling rate is straightforward and simple (see *e.g.* Landau and Lifshitz, 1968).

In our problem we are dealing with the southern and northern domains of generation separated by the equatorial region in which magnetic field generation is weak. This equatorial region corresponds to the barrier. Neglecting the interaction of the dynamo waves generated in different hemispheres, one can obtain the WKB eigenfunctions and eigenvalues explicitly [see Eqs.(9—15)]. Using the solution in the northern hemisphere $\vec{n}(\theta)$ and in the southern

one $\vec{s}(\theta)$, we follow the classical Lifshitz solution of the quantum problem and construct the dipole and quadrupole eigenfunctions as follows

$$\vec{d}(\theta) = \frac{1}{\sqrt{2}}\left[\vec{n}(\theta) + \vec{s}(\theta)\right] \tag{28}$$

and

$$\vec{q}(\theta) = \frac{1}{\sqrt{2}}\left[\vec{n}(\theta) - \vec{s}(\theta)\right], \tag{29}$$

where factor $1/\sqrt{2}$ is introduced in order to preserve the norms of the eigenvectors.

Let us note, that the solution in the southern hemisphere can be obtained easily by writing

$$\vec{s}(\theta) = \hat{P}\hat{U}\vec{n}(\theta).$$

To proceed further, let us introduce the following "inner product" of two vector functions $\vec{f}$ and $\vec{g}$:

$$\left(\vec{f}, \vec{g}\right) = \int\limits_0^{\pi/2} \left[f_1(\theta)g_1^*(\theta) + f_2(\theta)g_2^*(\theta)\right] \cos\theta d\theta, \tag{30}$$

where

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{and} \quad \vec{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

By multiplying equation $\hat{\mathcal{H}}\vec{n} = \gamma_0\vec{n}$ on $\vec{d}^a$ and Eq.(27) on $\vec{n}$, we obtain

$$\gamma_d^* - \gamma_0 = \frac{\left(\left(\hat{\mathcal{H}}^\dagger \vec{d}^a\right)^*, \vec{n}\right) - \left(\vec{d}^{a*}, \hat{\mathcal{H}}\vec{n}\right)}{\left(\vec{d}^{a*}, \vec{n}\right)} \tag{31}$$

and

$$\gamma_q^* - \gamma_0 = \frac{\left(\left(\hat{\mathcal{H}}^\dagger \vec{q}^a\right)^*, \vec{n}\right) - \left(\vec{q}^{a*}, \hat{\mathcal{H}}\vec{n}\right)}{\left(\vec{q}^{a*}, \vec{n}\right)}. \tag{32}$$

Let us note here that the dynamo-wave generated in the southern hemisphere $\vec{s}(\theta)$, if exists, is exponentially small in the northern one. Thus, we conclude:

$$\left|\left(\vec{s}^{a*}, \vec{n}\right)\right| \ll \left|\left(\vec{n}^{a*}, \vec{n}\right)\right|.$$

Hence

$$\left(\vec{d}^{a*}, \vec{n}\right) \sim \left(\vec{q}^{a*}, \vec{n}\right) \sim \left(\vec{n}^{a*}, \vec{n}\right)$$

With a good accuracy, we can neglect the corresponding difference in the denominators of expressions (31) and (32) and obtain the following important identity:

$$\gamma_d^* - \gamma_q^* = 2\frac{\left(\left(\hat{\mathcal{H}}^\dagger \vec{s}^a\right)^*, \vec{n}\right) - \left(\vec{s}^a, \hat{\mathcal{H}}\vec{n}\right)}{(\vec{n}^{a*}, \vec{n})}. \tag{33}$$

This equation brings the dynamo problem into direct correspondence with the quantum one.

Using the explicit expressions for the operators $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^\dagger$, we obtain the following relation:

$$\frac{1}{2}(\vec{n}^{a*}, \vec{n})\,\Delta\gamma^* = \int\limits_0^{\pi/2} d\theta \cos\theta \Big[\Delta s_1^a n_1 + D\frac{d}{d\theta}\left(\cos\theta\, s_2^a\right) n_1 + \Delta s_2^a n_2$$

$$-s_1^a\,(\Delta n_1) + D s_2^a \frac{d}{d\theta}\left(\cos\theta\, n_1\right) - s_1^a\,(\Delta n_2)\Big], \tag{34}$$

where indexes "1" and "2" correspond to the upper and lower components of a vector and s^a means the solution of the adjoint equation in the southern hemisphere. Integral Eq.(34) can be easily evaluated by parts and the splitting is expressed as boundary terms.

$$\frac{1}{2}(\vec{n}^{a*}, \vec{n})\,\Delta\gamma^* = n_1\frac{ds_1^a}{d\theta} + n_2\frac{ds_2^a}{d\theta} - s_1^a\frac{dn_1}{d\theta} - s_2^a\frac{dn_2}{d\theta} + Dn_1 s_2^a\Big|_{\theta=0} \tag{35}$$

5 Eigenvalue splitting

In this section we calculate the eigenvalue splitting explicitly using the WKB solution obtained previously (see Sec. 2). First of all, we find the eigensolution of the adjoint operator (8).

Let vector $\vec{n}(\theta)$, with the upper component $n_1(\theta)$ and lower $n_2(\theta)$, be the WKB solution of Eq.(5) [see Eqs. (9—15)]. Then, the corresponding solution of the adjoint equation can be obtained by the following replacements:

$$n_{1,2} \to n_{2,1},$$

$$\Gamma_0 \to \Gamma_0^* \quad \text{and} \quad \Gamma_1 \to \Gamma_1^*.$$

Function $k(\theta)$ satisfying the Hamilton-Jakobi equation (12) should be replaced by the following value:

$$k(\theta) \to -k^*(\theta).$$

After the appropriate replacements are made, the adjoint solution takes the form:

$$\vec{n}^a(\theta) = \begin{pmatrix} -i\varepsilon^{-2}k^*(\theta)\cos\theta \\ \Gamma_0^* + k^*(\theta)^2 \end{pmatrix} \frac{\sigma^*(\theta)}{\cos\theta} \exp\left[-\frac{iS^*(\theta)}{\varepsilon}\right], \tag{36}$$

At this point, we can calculate the eigenvalue splitting using the explicit expressions obtained above. We start with the evaluation of the inner product $(\vec{n}^a\,,\,\vec{n})$. Let us note, that in the double-well problem the corresponding product is nothing but the norm of the wave-function and, thus, is equal to unity. Using Eqs.(11) and (36) we get:

$$(\vec{n}^a\,,\,\vec{n}) = \frac{1}{\varepsilon^2}\int\limits_0^{\pi/2} d\theta \quad \left[-\left(\Gamma_0 + k^2(\theta)\right) ik^*(\theta) + ik(\theta)\left(\Gamma_0 + k^2(\theta)\right)^*\right]$$

$$\times \quad |\sigma(\theta)|^2 \exp\left\{-\frac{2}{\varepsilon}\int\limits_0^{\theta} \mathrm{Im}\, k(\theta')d\theta'\right\}. \tag{37}$$

Since the integrand in the above expression contains a Gaussian exponent with a sharp maximum, it is possible to evaluate the integral by the Laplace method. In the saddle point approximation, we obtain

$$(\vec{n}^{a*}\,,\,\vec{n}) = \frac{4\mathrm{Im}\,\Gamma_0}{\varepsilon^2}\sqrt{\frac{\pi\varepsilon}{\mathrm{Im}\,k'(\theta_*)}}\,|\sigma(\theta_*)|^2\,k(\theta_*)\exp\left\{-\frac{2}{\varepsilon}\int\limits_0^{\theta_*}\mathrm{Im}\,k(\theta)d\theta\right\}, \tag{38}$$

where θ_* is the point where action $S(\theta)$ reaches its maximum, *i.e.* the point at which $\mathrm{Im}\,k(\theta_*) = 0$. This point has already been found explicitly by Kuzanyan and Sokoloff (1995) as follows

$$\frac{\tilde{\alpha}(\theta_*)}{\tilde{\alpha}_{\max}} = \frac{9\sqrt{3}}{16\sqrt{2}\sqrt{\sqrt{3}-1}} \approx 0.8052. \tag{39}$$

Let us note that quantity (38) is a real, positive and exponentially large number. Thus, the eigenvalue splitting is exponentially small which is in accord with the familiar result for the double well problem.

Using expressions (19)–(22), we obtain the following results for boundary terms (35):

$$\left[n_1\frac{ds_1^a}{d\theta} - s_2^a\frac{dn_2}{d\theta}\right]\Bigg|_{\theta=0} = \frac{2ab}{\varepsilon^3}\mathrm{Re}\,\left[\mathrm{e}^{(-i\pi/6)}\sqrt{\Gamma_0}\right], \tag{40}$$

$$\left[n_2 \frac{ds_2^a}{d\theta} - s_1^a \frac{dn_1}{d\theta} \right] \Bigg|_{\theta=0} = \frac{2ab}{\varepsilon^3} \mathrm{Re} \left[\mathrm{e}^{(-i\pi/6)} \sqrt{\Gamma_0} \right], \tag{41}$$

$$D n_1 s_2^a \Big|_{\theta=0} = - \frac{a^2 \sqrt{\theta_1}}{\varepsilon^3} \mathrm{e}^{(-i\pi/3)}. \tag{42}$$

The two first boundary terms give real positive contributions to the splitting. This is not very surprising, since the corresponding terms come from the hermitian part of operator $\hat{\mathcal{H}}$. What is more remarkable is that the matching point θ_1 drops out of the final result for these terms. The third "nonhermitian" term gives a nonvanishing contribution to the imaginary part of the eigenvalue difference and explicitly depends upon the matching point. Since $\theta_1 \sim \varepsilon$, we see that this contribution is parametrically small compared to the first two ones.

Using Eqs.(40, 41, 42) we obtain the following result:

$$(\vec{n}^a\,,\,\vec{n})\,\mathrm{Re}\,\Delta\gamma = \frac{1}{\varepsilon^3}\,\frac{3^{9/4}}{2^{1/3}}\,\tilde{\alpha}_{\max}^{4/3}\,\sqrt{\tilde{\alpha}'(0)} \tag{43}$$

and

$$(\vec{n}^a\,,\,\vec{n})\,\mathrm{Im}\,\Delta\gamma = -\frac{\sqrt{\varepsilon}}{\varepsilon^3}\,2^{19/6}3^{1/2}\sqrt{\frac{\theta_1}{\varepsilon}}\,\tilde{\alpha}_{\max}^{4/3}\,\tilde{\alpha}'(0), \tag{44}$$

where norm $(\vec{n}^{a*}\,,\,\vec{n})$ was calculated in Sec. 4 [see Eq. (38)]. The matching point θ_1 can be calculated self-consistently as a point at which the phase-shifts for different asymptotics coincide [*cf.* Galitski and Sokoloff (1999)].

As we have seen above, the WKB approximation breaks down near the equator. This happens because $\theta = 0$ is the turning point for our solution. Thus, straightforward evaluation of (35) using the WKB formulae is not possible. Note that in the usual quantum problem such a difficulty does not arise since the boundary is located far from the turning point. The WKB solution can be applied directly and the exponential term coefficient can be found easily. However, it is important to realize that the method itself is limited by the exponential accuracy. The numerical part of the exponential term coefficient should not be trusted even if found. There are other asymptotical methods allowing the calculation of the tunneling rate, which give slightly different results. Indeed, the exponential factor is the same for all these methods. For a detailed discussion of the asymptotical methods in quantum double-well problems see *e.g.* Cooper *et al.* (1995). However, as we have already mentioned, the method used does not allow us to obtain the correct value of the exponential term coefficient. We conclude that the

accuracy of Eqs.(43, 44) already exceeds the accuracy of the method. Hence, the exact value of $\theta_1 \sim \varepsilon$ is not important.

Bearing these observations in mind, let us write the final result in the following form:

$$\frac{\mathrm{Re}\,\Delta\gamma}{|\gamma_0|} \approx \sqrt{\varepsilon}\exp\left\{\frac{2}{\varepsilon}\int_0^{\theta_*} |\mathrm{Im}\,k(\theta)|\,d\theta\right\} \tag{45}$$

and

$$\frac{\mathrm{Im}\,\Delta\gamma}{|\gamma_0|} \approx -\varepsilon\exp\left\{-\frac{2}{\varepsilon}\int_0^{\theta_*} |\mathrm{Im}\,k(\theta)|\,d\theta\right\} \tag{46}$$

6 Discussion

The overall picture of the magnetic field in the Parker model can be now summarized as follows. A strong dynamo wave appears at middle latitudes. The point where the generation sources reach maximum is shifted from the point where the helicity coefficient is maximal. This happens because the differential rotation and helicity mechanisms are both important on equal footing. In the framework of the asymptotic analysis, the two mechanisms manifest themselves through a universal function. The asymptotic WKB theory allows to derive explicit expressions describing the generated dynamo waves. It appears that at middle latitudes the main wave propagates equatorwards. At very high latitudes the wave changes its direction of motion and propagates towards the pole and reflects from it. In the vicinity of the equator there is a transition zone $\theta \sim \epsilon$ where the WKB solution becomes progressively worse and the crossover from the asymptotic behaviour (9) to (18) occurs. The wave does not vanish completely at the equator but rolls over it propagating in the southern part. Beyond the southern transition zone $\theta \sim -\varepsilon$, the WKB solution becomes applicable again and it describes a very weak decaying dynamo wave.

Let us mention here the observational results which refer to the time of the Maunder minimum (see Sokoloff and Nesme-Ribes, 1994; Ribes and Nesme-Ribes, 1993). During the Maunder minimum, the solar magnetic activity was very weak. At the end of the minimum, some signs of the activity appeared in the southern hemisphere. The southern wave of magnetic activity looked normal and propagated equatorwards. In the northern part there was almost no activity at all, except an unusual wave which existed only near the equator and propagated away from it.

Certainly, our simple linear asymptotic theory can not suggest any explanations for the Maunder minimum itself. However, we think that the weak equatorial wave detected at the epoch of the minimum corresponds to the equatorial wave which appears naturally in the framework of our asymptotic analysis. On the modern Sun, this equatorial wave, if exists, is screened by the background of the main dynamo wave. There are observational signatures of some magnetic field tracers such as extreme ultraviolet (EUV) lines to propagate polewards near the solar equator (*e.g.*, Benevolenskaya et al., 2001). They may indicate the presence of such a straying wave propagating from the other hemisphere. Notice, that the analysis of the observational data by Harvey (1992) revealed that the reversed polarity active regions that violate the Hale's polarity law are located mainly at low latitudes. Further observations of the equatorial region would be highly desirable to clarify this question.

We have shown that a small interaction between the northern and southern waves yields a splitting of the eigenvalues corresponding to the dipole and quadrupole configurations. It appears that the growth rate corresponding to the dipole configuration exceeds the quadrupole one. This fact gives a very tentative indication that the dipole configuration is to dominate the quadrupole one after a sufficiently long time. Let us estimate this time using the following trial parameters: $\alpha(\theta) = \sin\theta$ and $D = -10^3$. In this case

$$\frac{\mathrm{Re}\,\Delta\gamma}{|\gamma_0|} \sim 0.03$$

If we suppose that $(2\pi/\mathrm{Im}\,\gamma_0) \approx 22\,\mathrm{yr}$, where $\mathrm{Im}\,\gamma_0 = |\gamma_0|\sqrt{3}/2$. Then we have the following estimate for the time of dominance of the dipole mode over the quadrupole one $\tau = 1/\mathrm{Re}\,\Delta\gamma \sim 100\,\mathrm{yr}$. This is comparable with the time of recovery of the solar magnetic field generation (Ribes and Nesme-Ribes, 1993) from the Maunder Minumum when some activity was confined to one southern hemisphere (equatorial asymmetry).

The difference in the imaginary part of the eigenvalues may give rise to an additional weak period of oscillations of the magnetic field in the non-linear regime. The corresponding estimate for the period of such a global modulation is $T_2 = 2\pi/\mathrm{Im}\,\Delta\gamma \sim 2 \cdot 10^3\,\mathrm{yr}$.

Let us stress that the applicability of the simple Parkler's migratory dynamo model is limited. A two-dimensional and non-linear generalizations are required to provide a more accurate description. However, some simple estimates can be done now. Namely, using a two-dimensional asymptotic solution, *e.g.*, obtained recently by Belvedere *et al.* (2000), we can estimate

the overlap of the dynamo waves generated in different hemispheres. Because the r.h.s. of equation (34) then turns to a 2D integral, we expect that the corresponding overlap exceeds significantly that one in the one-dimensional model. This may result in a larger splitting of the eigenvalues. The corresponding period of secondary oscillations T_2 should increase. This qualitative result suggests a possible explanation for the well-known Gleissberg cycle.

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